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Dynamics of a charged particle in a linearly polarized traveling wave

Hamiltonian approach to laser-matter interaction at very high intensities

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Abstract. The basic physical processes in laser-matter interaction, up to 10^{17} W/cm^2 (for a neodymium laser) are now well understood, on the other hand, new phenomena evidenced in PIC code simulations have to be investigated above 10^{18} W/cm^2. Thus, the relativistic motion of a charged particle in a linearly polarized homogeneous electromagnetic wave is studied, here, using the Hamiltonian formalism. First, the motion of a single particle in a linearly polarized traveling wave propagating in a non-magnetized space is explored. The problem is shown to be integrable. The results obtained are compared to those derived considering a cold electron plasma model. When the phase velocity is close to c, it is shown that the two approaches are in good agreement during a finite time. After this short time, when the plasma response is taken into account no chaos take place at least when considering low densities and/or high wave intensities. The case of a charged particle in a traveling wave propagating along a constant homogeneous magnetic field is then considered. The problem is shown to be integrable when the wave propagates in vacuum. The existence of a synchronous solution is shown very simply. In the case when the wave propagates in a low density plasma, using a simplifying Lorentz transformation, it is shown that the system can be reduced to a time-dependent system with two degrees of freedom. The system is shown to be nonintegrable, chaos appears when a secondary resonance and a primary resonance overlap. Finally, stochastic instabilities are studied by considering the motion of one particle in a very high intensity wave perturbed by one or two low intensity traveling waves. Resonances are identified and conditions for resonance overlap are studied.

PACS. 52.38.Kd Laser-plasma acceleration of electrons and ions – 02.50.Ey Stochastic processes

1 Introduction

A large number of issues remain open in the study of laser-matter interaction at very high intensities. Recently, particle-in-cell (PIC) code simulations published by Tajima, Kishimoto, and Masaki have shown that the irradiation of very high intensity lasers on clustered matter leads to a very efficient heating of electrons [1,2]. They show that chaos seems to be the origin of the strong laser coupling with clusters. More recently, it was confirmed in PIC code simulations, in the case of two counterpropagating laser pulses, that stochastic heating can lead to an efficient acceleration of electrons [3,4]. Consequently, we started to explore situations, where electron trajectories become chaotic and how chaos can play an important part in laser-matter interaction at very high intensities. Therefore, the issue that we will address below is the stability of electron motion in the fields of the wave.

This paper is devoted to situations when the intensity of the wave is very high and/or when the plasma has a very low density compared to the non-relativistic critical density. A large part of this paper concerns situations when the polarization current is small compared to the displacement current $1/c^2\ (\partial E/\partial t) \gg \mu_0 nev$ [5]. During a short time, it can be considered that the electromagnetic wave propagates in vacuum. After this short time the plasma response has to be taken into account and can be treated as a perturbation. In most situations investigated in this paper, the dynamics of an electron alone in the wave will be studied first, and then the effect of the plasma response will be explored. As the PIC code simulations results previously published evidence a stochastic electron heating, due to a plane wave effect, the plane wave approximation is used. Nethertheless, the ponderomotive effects are not ignored in the sense that they contribute to make the wave interact with a low-density plasma.

At very high intensities the motion of a charged particle in a wave is highly non-linear. The situations when the

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motion is integrable are exceptional. The solutions corresponding to these situations deserve to be studied because they are strong as predicted by KAM theorem [6–11]. Moreover, they are useful to predict resonances when a perturbing wave is considered.

The paper is organized as follows: in Section 2, the dynamics of one particle in a linearly polarized traveling wave propagating in a nonmagnetized medium is studied; a certain number of results obtained previously are recalled. They are necessary to make the paper easy to read. First, the medium is assumed to be vacuum. The Liouville integrability of the motion is demonstrated [6–10,12,13]. Using a canonical transformation eliminates time; the new system is shown to be completely integrable. This demonstrates that trajectories are on a torus and that the state of the system can be expressed in terms of canonical action-angle variables [6–10,14]. Solving Hamilton-Jacobi equation also proves integrability. It is shown that the charged particle can have a high average velocity along the direction of propagation of the wave and along its electric field. These results are used to compare this one particle model to a plasma approach. Indeed, it is shown that a strong linearly polarized electromagnetic wave propagating in cold electron plasma, can generate a constant electron current along its direction of propagation during a short time. This time grows as the phase velocity of the wave becomes closer to the speed of light [15]. The results obtained with the two approaches are in good agreement during the short time. After this time, the plasma response has to be taken into account. It is shown that, even then, the motion of one particle of the plasma remains integrable in the limit of low densities.

The dynamics of a charged particle in a linearly polarized electromagnetic wave propagating along a constant homogeneous magnetic field is studied next in Section 3. In a first part the electromagnetic wave is assumed to propagate in vacuum. Roberts and Buchsbaum have already explored this problem in the case of a circularly polarized wave [16]. They found a "synchronous" solution in which the particle gains energy indefinitely. It is shown that the synchronous solution still exists when the wave is linearly polarized [15,17,18]. One of the constants of motion appears in the resonance condition [19], this means that when a particle is initially resonant it remains resonant forever. Two constants, which are canonically conjugate, are found. This property is used to reduce the initially three degrees of freedom problem to a two degree of freedom problem. The system is integrated and is shown to be Liouville integrable. An asymptotic solution for the energy, and consequently all of the variables of the system, is found.

Then, in order to study the plasma response in the case of a very high intensity wave propagating in a low-density plasma, an approximate solution for an almost transverse wave is used. It is shown that for moderate values of the constant magnetic fields one can consider that this wave has approximately the same form as the one, which propagates, in vacuum. Performing a Lorentz transformation eliminates the space variable corresponding to the direction of propagation of the wave. Just like in the previous case, two canonically conjugate constants are used to reduce the initially three degrees of freedom problem to a two degrees of freedom problem. Thus, Poincaré maps are performed. Lyapunov exponents are also calculated to confirm the chaotic nature of some trajectories [9,10,20]. Chaos appears when a secondary resonance and a primary resonance overlap. Consequently, the system is not integrable and chaos appears as soon as the plasma response is taken into account. The overlap of the two resonances can generate some stochastic heating. Still, more stochastic heating should be generated when many resonances can overlap, such a situation can be achieved when perturbing the high intensity wave by a low intensity electromagnetic traveling plane wave. This is discussed in the next section.

In Section 4, the stability of a charged particle in the fields of many waves is explored. In this part no constant homogeneous magnetic field will be taken into account. A high intensity plane wave will be first perturbed by one or two electromagnetic plane waves. Then, a perturbing plasma wave is considered. The solution of Hamilton-Jacobi equation derived in Section 2 is used to identify resonances. The Chirikov criterion [21] is applied to two resonances corresponding to two symmetric perturbing waves. In this case, the particle trajectory is assumed not to be affected by the plasma response, as the medium is not magnetized. In the case of a plasma wave, the effect of one or two perturbing modes is investigated. Above the Chirikov threshold, computing single particle trajectories with their initial conditions in the overlapping region, calculating their energy evidences stochastic heating. In each situation, the Chirikov criterion is applied to the two most dangerous resonances.

2 Dynamics of a charged particle in an electromagnetic linearly polarized traveling wave

2.1 The wave propagates in vacuum

2.1.1 Hamiltonian formulation of the problem. Integrability of the system

Let us consider a charged particle in an electromagnetic plane wave propagating along the z-direction (the wave vector $\mathbf{k}_0$ is parallel to the z-direction). The following 4-potential is chosen

$$[\phi, \mathbf{A}] = \left[0, \frac{E_0}{\omega_0} \cos(\omega_0 t - k_0 z)\,\widehat{\mathbf{e}}_x\right], \tag{1}$$

where E_0, ω_0, and k_0 are constants.

When time t is treated as a parameter entirely distinct from the spatial coordinates, the relativistic Hamiltonian for a charged particle in the wave is in mks units

$$H = \left[\left[P_x + \frac{eE_0}{\omega_0} \cos(\omega_0 t - k_0 z) \right]^2 c^2 + P_y^2 c^2 + P_z^2 c^2 + m^2 c^4 \right]^{1/2}, \quad (2)$$

where $-e, m$, and $P_i (i = x, y, z)$ are respectively the particle's charge, its rest mass and its canonical momentum components. This system has three degrees of freedom. We next introduce the following dimensionless variables and parameter

$$\widehat{z} = k_0 z, \quad \widehat{P}_{x,y,z} = \frac{P_{x,y,z}}{mc}, \quad \widehat{t} = \omega_0 t,$$
$$\widehat{H} = \gamma = \frac{H}{mc^2}, \quad a = \frac{eE_0}{mc\omega_0}, \quad (3)$$

and perform the canonical transformation, $(\widehat{z}, \widehat{P}_z) \rightarrow (\zeta, \widehat{P}_z)$, given by the type-2 generating function [6–10,15,17]

$$F_2\left(\widehat{z}, \widehat{P}_z\right) = \widehat{P}_z\left(\widehat{z} - \widehat{t}\right). \quad (4)$$

This canonical transformation keeps $\widehat{P}_z$ unchanged and yields

$$\zeta = \widehat{z} - \widehat{t}. \quad (5)$$

The Hamiltonian expressed in terms of the new variables is

$$\widehat{H} = \widehat{C} = \left[\left(\widehat{P}_x + a \cos\zeta \right)^2 + \widehat{P}_y^2 + \widehat{P}_z^2 + 1 \right]^{1/2} - \widehat{P}_z. \quad (6)$$

This Hamiltonian (we have also called it $\widehat{C}$ as this constant of motion which is associated to the plane wave symmetry will be named this way further), $\widehat{P}_x$ and $\widehat{P}_y$ are three constants of motion, which are independent and in involution. As a consequence the system is completely integrable [6–10].

2.1.2 Integration of the Hamilton-Jacobi equation

When using the proper time of the particle to parametrize the motion in the extended phase space, the Hamiltonian of the charged particle in the wave reads [22]

$$H = \frac{1}{2} mc^2 \gamma^2 - \frac{1}{2m} (\mathbf{P} + \mathbf{eA})^2 - \frac{1}{2} mc^2. \quad (7)$$

When the dimensionless variables and parameter defined by equations (3) are used again, a normalized proper time: $\widehat{\tau} = \omega_0 \tau$ is also introduced. Then the normalized Hamiltonian reads

$$\widehat{H} = \frac{1}{2}\gamma^2 - \frac{1}{2}\left(\widehat{\mathbf{P}} + \mathbf{a}\right)^2 - \frac{1}{2}. \quad (8)$$

Although the electron motion is not restricted to the plane of polarization of the wave (the y degree of freedom is assumed to be excited), we look for a set of actions $(P_\perp, P_\parallel, E)$ and angles (θ, φ, ϕ), instead of the configuration $(\mathbf{r}, t)$, and momentum, $(\mathbf{P}, -\gamma)$ in the $(\widehat{x}, \widehat{z}, \widehat{t}\, \widehat{P}_x, \widehat{P}_z, -\gamma)$ phase space. This comes out to say that we seek a canonical transformation $(\widehat{x}, \widehat{z}, \widehat{t}, \widehat{P}_x, \widehat{P}_z, -\gamma) \rightarrow (\theta, \varphi, \phi, P_\parallel, P_\perp, E)$, such that the new momenta are constants of motion. Following Landau and Lifshitz [23], the following type-2 generating function is obtained [24]

$$\widehat{F}_2\left(P_\perp, P_\parallel, E, \widehat{x}, \widehat{z}, \widehat{t}\right) = P_\parallel \widehat{z} + P_\perp \widehat{x} - E\widehat{t} + \frac{P_\perp a}{P_\parallel - E} \sin\left(\widehat{t} - \widehat{z}\right) + \frac{a^2}{8P_\parallel - 8E} \sin 2\left(\widehat{t} - \widehat{z}\right). \quad (9)$$

The old configuration variables expressed in terms of the new ones are given by

$$\widehat{p}_x = P_\perp + a\cos(\phi + \varphi),$$
$$\widehat{p}_z = P_\parallel - \frac{P_\perp a}{P_\parallel - E}\cos(\phi + \varphi) - \frac{a^2}{4\left(P_\parallel - E\right)}\cos 2(\phi + \varphi),$$
$$\gamma = E - \frac{P_\perp a}{P_\parallel - E}\cos(\phi + \varphi) - \frac{a^2}{4\left(P_\parallel - E\right)}\cos 2(\phi + \varphi),$$
$$\widehat{x} = \theta + \frac{a}{P_\parallel - E}\sin(\phi + \varphi),$$
$$\widehat{z} = \varphi - \frac{P_\perp a}{\left(P_\parallel - E\right)^2}\sin(\phi + \varphi) - \frac{a^2}{8\left(P_\parallel - E\right)^2}\sin 2(\phi + \varphi),$$
$$\widehat{t} = -\phi - \frac{P_\perp a}{\left(P_\parallel - E\right)^2}\sin(\phi + \varphi) - \frac{a^2}{8\left(P_\parallel - E\right)^2}\sin 2(\phi + \varphi), \quad (10)$$

where $\widehat{p}_x$ and $\widehat{p}_z$ are components of the normalized momentum of the charged particle. It is straightforward to check that, $P_\perp = \langle \widehat{p}_x \rangle$, $P_\parallel = \langle \widehat{p}_z \rangle$, $E = \langle \gamma \rangle$, and $\widehat{C} = E - P_\parallel$. All the former constants of motion are found again. A term corresponding to the identity transformation in the $(\widehat{y}, \widehat{P}_y)$ plane, could have been added to the generating function (9). The new Hamiltonian in terms of the action variables reads [24]

$$\widetilde{H}_0\left(P_\parallel, P_\perp, \widehat{P}_y, E\right) = -\frac{1}{2}\left(M^2 + P_\parallel^2 + P_\perp^2 + \widehat{P}_y^2 - E^2\right), \quad (11)$$

where $M^2 = 1 + a^2/2$. As $\widetilde{H}_0 = 0$, the energy momentum dispersion relation is given by

$$E\left(P_\perp, \widehat{P}_y, P_\parallel, a\right) = \sqrt{M^2 + P_\perp^2 + \widehat{P}_y^2 + P_\parallel^2}. \quad (12)$$

The solution of Hamilton equations is

$$\theta = -P_\perp \widehat{\tau}, \quad \widehat{y} = -\widehat{P}_y \widehat{\tau}, \quad \varphi = -P_\parallel \widehat{\tau}, \quad \phi = E\widehat{\tau}. \quad (13)$$

Finding the solution of Hamilton-Jacobi equation is another way to prove that the problem is integrable. This solution is also very useful to predict resonances when a perturbing mode is considered.

2.2 The wave propagates in a cold relativistic electron plasma

We now show that the theory of propagation of a strong electromagnetic wave in a cold, low density, electron plasma, allows finding many of the results previously found considering one particle only. The good agreement between the two approaches last a finite time only. This time grows with the intensity of the wave and when the plasma density decreases. For instance, we are going to show that, during the finite time, a strong linearly polarized electromagnetic wave generates a constant current along its propagation direction when propagating in cold electron plasma with a phase velocity very close to the speed of light.

Following Akhiezer and Polovin [25] the propagation of a relativistically strong wave in a cold relativistic electron plasma is described with Maxwell and Lorentz equations. All the variables entering into these equations are assumed not to be functions of space and time separately, but only of the combination $\mathbf{i}\cdot\mathbf{r}-Vt$, where $\mathbf{i}$ is a constant unit vector, and V a constant. The meaning of this type of solution is that it represents plane waves traveling in the direction $\mathbf{i}$ with speed V. It is assumed that $\mathbf{v}$ and $\mathbf{p}$ are respectively the velocity and mechanical momentum of the electrons, n is their density and N_0 is the one of ions. Combining Lorentz and Maxwell equations and introducing the following variables

$$\widehat{\mathbf{p}}=\frac{\mathbf{p}}{mc},\quad \widehat{\mathbf{v}}=\frac{\mathbf{v}}{c},\quad \tau=t-\frac{\mathbf{i}\cdot\mathbf{r}}{V},\quad \omega_p^2=\frac{e^2N_0}{\varepsilon_0 m}. \tag{14}$$

Letting $\theta=\omega_p\left(\beta^2-1\right)^{-1/2}\tau$, lead to the wave equations. In the absence of the external magnetic field $\mathbf{B}_0$, the following equations for the electron motion are found after some algebra [25]

$$\frac{d^2\widehat{p}_x}{d\theta^2}+\frac{\beta^3\widehat{p}_x}{\beta\sqrt{1+\widehat{p}^2}-\widehat{p}_z}=0, \tag{15a}$$

$$\frac{d^2\widehat{p}_y}{d\theta^2}+\frac{\beta^3\widehat{p}_y}{\beta\sqrt{1+\widehat{p}^2}-\widehat{p}_z}=0, \tag{15b}$$

$$\frac{d^2}{d\theta^2}\left(\beta\widehat{p}_z-\sqrt{1+\widehat{p}^2}\right)+\frac{\beta^2\left(\beta^2-1\right)\widehat{p}_z}{\beta\sqrt{1+\widehat{p}^2}-\widehat{p}_z}=0. \tag{15c}$$

where $\beta=V/c$.

Neglecting the last term in the third equation (Eq. (15c)) when the phase velocity is close to the speed of light, i.e. $\beta\approx 1$, we find

$$\widehat{C}=\alpha^2=\sqrt{1+\widehat{p}^2}-\widehat{p}_z, \tag{16}$$

where $\widehat{C}$ is the invariant already found in the one particle Hamiltonian approach (Eq. (6)).

By solving numerically equations (15), we have shown that there is a good agreement between the values calculated considering one particle only or a plasma during a finite time. This time increases when the intensity of the wave grows and when the density of the plasma decreases.

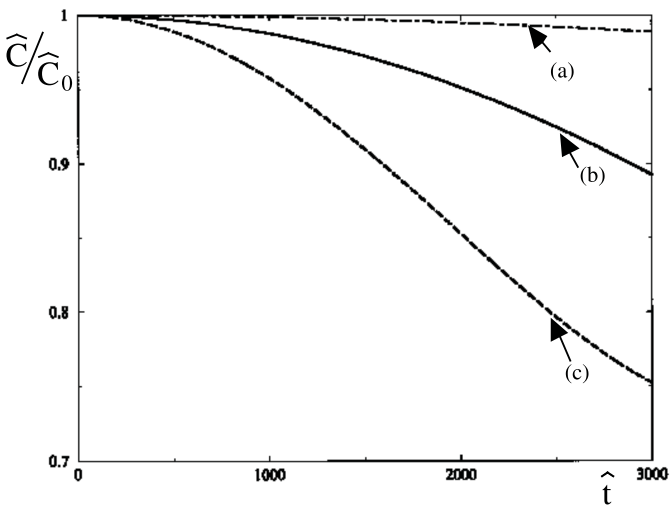


Fig. 1. Evolution of $\widehat{C}$ for different values of a when $\omega_p^2/\omega_0^2=10^{-3}$. (a) $a=100$, (b) $a=10$, (c) $a=1$.

Figure 1 shows the evolution of $\widehat{C}$ versus time for a given low density, it shows that $\widehat{C}$ is all the more a constant as the wave intensity is higher.

Considering that the electron motion is in the $x-z$-plane ($\widehat{p}_y=0$), assuming that $\beta\approx 1$ and $\widehat{p}_x=\widehat{p}_{x0}$ when $\theta=0$, an approximate solution of equations (15a, 15b) is given by

$$\widehat{p}_x=\widehat{p}_{x0}\cos\frac{\theta}{\alpha}, \tag{17a}$$

$$\widehat{p}_y=0. \tag{17b}$$

Substituting these equations into (16), we find

$$\widehat{p}_z=\frac{1}{4\alpha^2}\left[\widehat{p}_{x0}^2-2\left(\alpha^4-1\right)+\widehat{p}_{x0}^2\cos\frac{2\theta}{\alpha}\right]. \tag{17c}$$

Still considering a wave that propagates in the $x-z$ plane, two ways of determining the constant $\widehat{C}$ (or α^2) can be chosen. The first way consists in assuming that $\widehat{C}$ is a real constant of motion that is determined with the initial conditions. The second way consists in assuming that the average current in the direction of propagation of the wave vanishes [25]. The first way to determine $\widehat{C}$ leads to a solution that is accurate at very low densities and during a short time only. Moreover, during this short time, the average velocity along the direction of propagation of the wave is almost the same as the one calculated with the one particle model. The second way to determine $\widehat{C}$ gives a more accurate solution when considering long times and higher densities. The two solutions found this way for E_z show that the wave become transverse as the density goes to zero. In the two cases, the dynamics of an electron that is not a part of the background distribution can also be studied. P_x, P_y and $C=H-VP_z$ are still three constants independent and in involution. As a consequence the system is integrable. We can conclude that, in a low-density plasma, when considering waves polarized in a plane, only an almost linearly polarized wave can propagate; a small

E_z component has to be added to the electric field of the wave which propagates in vacuum. This component (E_z) does not generate chaos and no stochastic heating can take place.

3 Dynamics of a charged particle in a linearly polarized electromagnetic traveling wave propagating along a constant homogeneous magnetic field

3.1 The wave is assumed to propagate in vacuum

3.1.1 Hamiltonian structure of the problem

The constant magnetic field $\mathbf{B}_0$ is supposed to be along the z-axis. The traveling wave is assumed to be linearly polarized. It has a propagation vector $\mathbf{k}_0$ parallel to $\mathbf{B}_0$. The following vector potential is chosen for the electromagnetic field

$$\mathbf{A} = \left[-\frac{B_0}{2}y + \frac{E_0}{\omega_0}\cos(\omega_0 t - k_0 z)\right]\widehat{\mathbf{e}}_x + \left(\frac{B_0}{2}x\right)\widehat{\mathbf{e}}y. \quad (18)$$

The scalar potential is assumed to vanish. The relativistic Hamiltonian for the motion is

$$H = \left[\left(P_x + \frac{eE_0}{\omega_0}\cos(\omega_0 t - k_0 z) - \frac{eB_0}{2}y\right)^2 c^2 + \left(P_y + \frac{eB_0}{2}x\right)^2 c^2 + P_z^2 c^2 + m^2 c^4\right]^{\frac{1}{2}}. \quad (19)$$

This is a time-dependent system with three degrees of freedom. It can be easily checked that $C = H - (\omega_0/k_0)P_z$ is still a constant of motion for this system. Combining the equations of Hamilton allows us to find easily two constants of motion

$$C_1 = P_x + \frac{eB_0}{2}y, \quad C_2 = P_y - \frac{eB_0}{2}x. \quad (20)$$

These two constants are such that

$$\left[C_1, \frac{C_2}{eB_0}\right] = 1. \quad (21)$$

The three constants that we have just found are not in involution and, at this stage, one cannot conclude that the problem is integrable.

The normalized equations of motion are obtained by introducing new normalized variables $\widehat{x} = k_0 x, \widehat{y} = k_0 y$, and normalized parameter $\Omega_0 = eB_0/m\omega_0$ to those already defined by (3). The following two normalized constants are also introduced, $\widehat{C}_1 = C_1/mc$ and $\widehat{C}_2 = C_2/mc$.

3.1.2 Integrability of the problem

The canonical transformation $\zeta = \widehat{z} - \widehat{t}$, defined by the generating function (4) is performed first. The normalized Hamiltonian is given by

$$\mathbf{H} = \left[\left(\widehat{P}_x + a\cos\zeta - \frac{\Omega_0}{2}\widehat{y}\right)^2 + \left(\widehat{P}_y + \frac{\Omega_0}{2}\widehat{x}\right)^2 + \widehat{P}_z^2 + 1\right]^{1/2} - \widehat{P}_z. \quad (22)$$

The fact that the two constants $\widehat{C}_1$ and $\widehat{C}_2/\Omega_0$ are canonically conjugate is used now to reduce the system. To do so, a canonical transformation, which is the product of two canonical transformations, defined by the following type-2 generating functions: $F_2 = [\widetilde{P}_x - (\Omega_0/2)\widehat{y}]\widehat{x} + \widetilde{P}_y\widehat{y}$ and $F_2 = (P_2 + \Omega_0\tilde{x})\tilde{y} + P_1(\tilde{x} + P_2/\Omega_0)$, given by

$$\widehat{x} = Q_1 - \frac{P_2}{\Omega_0}, \qquad \widehat{y} = Q_2 - \frac{P_1}{\Omega_0},$$
$$\widehat{P}_x = \frac{1}{2}(\Omega_0 Q_2 + P_1), \quad \widehat{P}_y = \frac{1}{2}(\Omega_0 Q_1 + P_2), \quad (23)$$

is performed [17,26]. In terms of the new variables, the Hamiltonian is

$$H = \widehat{C} = \left[(P_1 + a\cos\zeta)^2 + \Omega_0^2 Q_1^2 + \widehat{P}_z^2 + 1\right]^{1/2} - \widehat{P}_z. \quad (24)$$

The equations of Hamilton are

$$\dot{P}_1 = -\frac{\Omega_0^2}{\gamma}Q_1, \qquad \dot{Q}_1 = \frac{1}{\gamma}(P_1 + a\cos\zeta),$$
$$\dot{\widehat{P}}_z = \frac{a}{\gamma}\sin\zeta\,(P_1 + a\cos\zeta), \qquad \dot{\zeta} = \frac{\widehat{P}_z}{\gamma} - 1. \quad (25)$$

The equation of Hamilton for ζ (Eqs. (25)), can be put in the form

$$\gamma\frac{d\zeta}{d\widehat{t}} = -\widehat{C}. \quad (26)$$

As a consequence, indicating differentiation with respect to ζ by a prime in this paragraph, we can write

$$A' = \frac{dA}{d\zeta} = \frac{dA/d\widehat{t}}{d\zeta/d\widehat{t}}, \quad (27)$$

which implies that $\dot{A} = -A'\widehat{C}/\gamma$. Thus, the equations of Hamilton (Eqs. (25)) become

$$P_1'\widehat{C} = \Omega_0^2 Q_1, \quad (28a)$$
$$Q_1'\widehat{C} = -(P_1 + a\cos\zeta), \quad (28b)$$
$$P_z'\widehat{C} = -a\sin\zeta\,(P_1 + a\cos\zeta), \quad (28c)$$
$$\widehat{C} = \gamma - P_z. \quad (28d)$$

Differentiating a second time the second equation leads to the following equations of motion for Q_1

$$Q_1'' + \frac{\Omega_0^2}{\widehat{C}^2}Q_1 = \frac{a}{\widehat{C}}\sin\zeta. \quad (29a)$$

The following equation for P_1, is obtained in the same way

$$P_1'' + \frac{\Omega_0^2}{\widehat{C}^2} P_1 = -\frac{\Omega_0^2}{\widehat{C}^2} a \cos\zeta. \tag{29b}$$

One has a resonance when $\Omega_0^2 = \widehat{C}^2$. These equations can be easily solved analytically when the resonance condition is satisfied and when it is not. Then equation (28c) is used to determine P_z and γ is obtained through equation (28d). The Lorentz factor is given by: $\gamma = \Omega_0 + P_z$

$$\gamma = \Omega_0 + \frac{a^2}{8\Omega_0}\zeta^2 - \frac{a\bar{A}}{2}\sin\bar{\varphi}\zeta + \frac{a^2}{8\Omega_0}\zeta\sin 2\zeta + \frac{3a^2}{16\Omega_0}\cos 2\zeta + \frac{a\bar{A}}{4}\sin\left(2\zeta + \bar{\beta}\right) + \bar{\Lambda}, \tag{30}$$

where $\bar{A}$, $\bar{\Lambda}$, $\bar{\varphi}$ and $\bar{\beta}$ are arbitrary constants which can be obtained from the initial conditions. Let us point out that when ζ is large enough, γ has the following asymptotic expression

$$\gamma \approx \frac{a^2}{8\Omega_0}\zeta^2. \tag{31}$$

We have verified that the solution found this way is the same as the one found by Ondarza-Rovira [27].

As we have shown that the system can be integrated when the wave is assumed to propagate in vacuum, no chaos can take place and consequently trajectories are all regular.

Integrability can also be demonstrated by using Liouville's theorem. Another constant was found and used to test our numerical calculations (Appendix A).

3.2 The wave propagates in a cold electron plasma

In this part the influence of the plasma is taken into account. The wave is still assumed to propagate along a constant homogeneous magnetic field $\mathbf{B}_0$.

The equations describing non-linear waves propagating along a constant magnetic field in a relativistic electron plasma (Eqs. (15)) have to be replaced by the following [25]

$$\frac{d^2\widehat{p}_x}{d\theta^2} + \frac{\beta^3\widehat{p}_x}{\beta\sqrt{1+\widehat{p}^2} - \widehat{p}_z} = \beta\sqrt{(\beta^2-1)}\Omega_p \frac{d}{d\theta}\left(\frac{\widehat{p}_y}{\beta\sqrt{1+\widehat{p}^2} - \widehat{p}_z}\right), \tag{32a}$$

$$\frac{d^2\widehat{p}_y}{d\theta^2} + \frac{\beta^3\widehat{p}_y}{\beta\sqrt{1+\widehat{p}^2} - \widehat{p}_z} = -\beta\sqrt{(\beta^2-1)}\Omega_p \frac{d}{d\theta}\left(\frac{\widehat{p}_x}{\beta\sqrt{1+\widehat{p}^2} - \widehat{p}_z}\right), \tag{32b}$$

$$\frac{d^2}{d\theta^2}\left(\beta\widehat{p}_z - \sqrt{1+\widehat{p}^2}\right) + \frac{\beta^2\left(\beta^2-1\right)\widehat{p}_z}{\beta\sqrt{1+\widehat{p}^2} - \widehat{p}_z} = 0, \tag{32c}$$

where $\Omega_p = eB_0/m\omega_p$.

Assuming that the index of refraction of the medium is close to unity, equation (32c) still shows that, $\widehat{C} = \alpha^2 = \sqrt{1+\widehat{p}^2} - \widehat{p}_z$, is a constant. The first two equations (32) then take the form

$$\frac{d^2\widehat{p}_x}{d\theta^2} + \frac{\widehat{p}_x}{\alpha^2} = \sqrt{(\beta^2-1)}\frac{\Omega_p}{\alpha^2}\frac{d\widehat{p}_y}{d\theta}, \tag{33a}$$

$$\frac{d^2\widehat{p}_y}{d\theta^2} + \frac{\widehat{p}_y}{\alpha^2} = -\sqrt{(\beta^2-1)}\frac{\Omega_p}{\alpha^2}\frac{d\widehat{p}_x}{d\theta}. \tag{33b}$$

Introducing the following complex quantity $Z = \widehat{p}_x + i\widehat{p}_y$, these two equations are equivalent to the following

$$\frac{d^2 Z}{d\theta^2} + i\frac{\lambda}{\alpha^2}\frac{dZ}{d\theta} + \frac{1}{\alpha^2}Z = 0, \tag{34}$$

where $\lambda = \Omega_p\sqrt{\beta^2-1}$. The solution reads

$$Z = \Lambda_1 \exp i\left[\left(\frac{\sqrt{\lambda^2/\alpha^2+4}}{2\alpha} - \frac{\lambda}{2\alpha^2}\right)\theta + \varphi_1\right] + \Lambda_2 \exp -i\left[\left(\frac{\sqrt{\lambda^2/\alpha^2+4}}{2\alpha} + \frac{\lambda}{2\alpha^2}\right)\theta - \varphi_2\right], \tag{35}$$

where Λ_1, Λ_2, φ_1 and φ_2 are real constants which depend on the initial conditions.

Here again, it has been shown that, during a short time, the electrons of a plasma behave as if they were alone in the wave.

Looking for an approximate solution valid after a long time (a time longer than a plasma period), we assume that the average current along the z-axis vanishes, this implies that

$$\alpha^2 = \sqrt{1+\Lambda_1^2+\Lambda_2^2}. \tag{36}$$

Finally, using equation (35) and $\widehat{C}$, we obtain

$$\widehat{p}_x = \Lambda_1\cos\left(\omega_1\tau + \varphi_1\right) + \Lambda_2\cos\left(\omega_2\tau - \varphi_2\right), \tag{37a}$$

$$\widehat{p}_y = \Lambda_1\sin\left(\omega_1\tau + \varphi_1\right) - \Lambda_2\sin\left(\omega_2\tau - \varphi_2\right), \tag{37b}$$

$$\widehat{p}_z = \frac{\Lambda_1\Lambda_2}{\sqrt{1+\Lambda_1^2+\Lambda_2^2}}\cos\left[\left(\omega_1+\omega_2\right)\tau + \varphi_2 - \varphi_1\right], \tag{37c}$$

where

$$\omega_1 = \left[\frac{\sqrt{\frac{\Omega_p^2(\beta^2-1)}{\sqrt{1+\Lambda_1^2+\Lambda_2^2}}+4}}{2\left(1+\Lambda_1^2+\Lambda_2^2\right)^{1/4}} - \frac{\Omega_p\sqrt{\beta^2-1}}{2\sqrt{1+\Lambda_1^2+\Lambda_2^2}}\right]\omega_p\left(\beta^2-1\right)^{-1/2}, \tag{38a}$$

and

$$\omega_2 = \left[\frac{\sqrt{\frac{\Omega_p^2(\beta^2-1)}{\sqrt{1+\Lambda_1^2+\Lambda_2^2}}+4}}{2\left(1+\Lambda_1^2+\Lambda_2^2\right)^{1/4}} + \frac{\Omega_p\sqrt{\beta^2-1}}{2\sqrt{1+\Lambda_1^2+\Lambda_2^2}}\right]\omega_p\left(\beta^2-1\right)^{-1/2}. \tag{38b}$$

For low densities and weak magnetic fields, when $4\sqrt{1+\Lambda_1^2+\Lambda_2^2}/\Omega_p^2(\beta^2-1) \gg 1$ (we assume that Ω_p remains close to unity when $\beta - 1$ goes to zero), ω_1 goes to ω_2 and the solution that we have just found describes an almost monochromatic wave. When $\Lambda_1 = \Lambda_2$, this wave is also almost linearly polarized.

In this limit ($\beta \approx 1$), let us consider a test particle, which interacts with the fields of the wave without belonging to the main electron distribution. Thus, the wave vector potential has a form which can be approached by equations (18) where $\omega_0 \approx \omega_1 \approx \omega_2$. Just like before, dimensionless variables and parameters are used. The normalized Hamiltonian of the system expressed in terms of the dimensionless variables and parameters is

$$\widehat{H} = n\left[\left(\widehat{P}_x + a\cos\left(\widehat{t}-\widehat{z}\right) - \frac{\Omega_0}{2n}\widehat{y}\right)^2 + \left(\widehat{P}_y + \frac{\Omega_0}{2n}\widehat{x}\right)^2 + \widehat{P}_z^2 + 1\right]^{1/2}. \quad (39)$$

The Hamilton equations allow us to readily find two constants of motion

$$\widehat{C}_1 = \widehat{P}_x + \frac{\Omega_0}{2n}\widehat{y}, \qquad \widehat{C}_2 = \widehat{P}_y - \frac{\Omega_0}{2n}\widehat{x}. \quad (40)$$

It can be noted that the two constants $\widehat{C}_1$ and $n\widehat{C}_2/\Omega_0$ are canonically conjugate. $\widehat{C} = \widehat{H} - \widehat{P}_z$ is still a constant of motion. These three constants of motion are not in involution and one cannot conclude that the problem is integrable.

We have solved the equations of Hamilton numerically. When the wave propagates in a medium with an index of refraction inferior to unity ($n < 1$), the trajectories spiral outward and inward just as in the non resonant case when the wave propagates in vacuum.

3.2.1 Introduction of a simplifying Lorentz transformation

A new frame (L*) which moves uniformly along the z-axis with velocity U relative to the laboratory frame is introduced. The Lorentz transformation of the 4-momentum is given by [22,23]

$$P'_x = P_x, \quad P'_y = P_y, \quad P'_z = \Gamma\left(P_z - \frac{U}{c^2}E\right), \qquad E' = \Gamma\left(E - UP_z\right), \quad (41)$$

where $\Gamma = (1 - U^2/c^2)^{-1/2}$, and $E = \gamma mc^2$ is the energy of the charged particle.

In the extended phase space, where time is treated on a common basis with other coordinates, a fully covariant Hamiltonian formulation of the problem can be constructed. In this space, the Lorentz transformation defined above is identical to the canonical transformation generated by the following type-2 generating function

$$F_2\left(x,y,z,t,P'_x,P'_y,P'_z,E'\right) = P'_x x + P'_y y + \Gamma\left(P'_z + \frac{U}{c^2}E'\right)z - \Gamma\left(E' + UP'_z\right)t. \quad (42)$$

As a consequence, if the problem is integrable in the frame (L*), it is also integrable in the laboratory frame.

The phase of the wave which is an invariant takes, in the moving frame, the following form [22]

$$\omega_0 t - k_0 z = \Gamma\left[\omega_0\left(t' + \frac{U}{c^2}z'\right) - k_0\left(z' + Ut'\right)\right]. \quad (43)$$

When the phase velocity of the wave is greater than the speed of light, there exists one special frame (L*) in which the phase does not depend on the variable z'. This frame can be defined by its drift velocity [28]

$$\frac{U}{c} = \frac{k_0 c}{\omega_0} = n. \quad (44)$$

In (L*), the vector potential is [22]

$$\mathbf{A} = \left(-\frac{B_0}{2}y' + \frac{E'_0}{\omega'_0}\cos\omega'_0 t'\right)\widehat{\mathbf{e}}'_x + \left(\frac{B_0}{2}x'\right)\widehat{\mathbf{e}}'_y. \quad (45)$$

Within the new frame (L*), the equations of motion are generated by the following Hamiltonian

$$H' = \left[\left(P'_x + \frac{eE'_0}{\omega'_0}\cos\omega'_0 t' - \frac{eB_0}{2}y'\right)^2 c^2 + \left(P'_y + \frac{eB_0}{2}x'\right)^2 c^2 + P'_z c^2 + m^2c^4\right]^{1/2}. \quad (46)$$

Let us now introduce the following dimensionless variables and parameters

$$\widehat{x}' = \frac{\omega'_0}{c}x', \ \widehat{y}' = \frac{\omega'_0}{c}y', \ \widehat{z}' = \frac{\omega'_0}{c}z', \ \widehat{t}' = \omega'_0 t', \ \widehat{P}'_{x,y,z} = \frac{\widehat{P}'_{x,y,z}}{mc},$$
$$\Omega'_0 = \frac{eB_0}{m\omega'_0}, \ a' = \frac{eE'_0}{mc\omega'_0}, \ \widehat{H}' = \gamma' = \frac{H'}{mc^2}. \quad (47)$$

The following normalized Hamiltonian

$$\widehat{H}' = \left[\left(\widehat{P}'_x + a'\cos\widehat{t}' - \frac{\Omega'_0}{2}\widehat{y}'\right)^2 + \left(\widehat{P}'_y + \frac{\Omega'_0}{2}\widehat{x}'\right)^2 + \widehat{P}'^2_z + 1\right]^{1/2}, \quad (48)$$

leads to the normalized equations of motion.

Dropping the primes for convenience, it can be shown very easily that this system has three constants of motion

$$\widehat{C}_1 = \widehat{P}_x + \frac{\Omega_0}{2}\widehat{y}, \ \widehat{C}_2 = \widehat{P}_y - \frac{\Omega_0}{2}\widehat{x}, \ \widehat{C} = \widehat{P}_z. \quad (49)$$

The first two constants ($\widehat{C}_1$ and $\widehat{C}_2/\Omega_0$) are canonically conjugate.

3.2.2 Reduction to a two-dimensional problem in the new frame

Let us reduce the system in order to perform Poincaré maps. To do so, let us choose the two constants $\widehat{C}_1$ and $\widehat{C}_2$ as new momentum and coordinate conjugate. The canonical transformation defined by equations (23) is performed. The new Hamiltonian is

$$H = \left[\left(P_1 + a\cos\widehat{t}\right)^2 + \Omega_0^2 Q_1^2 + \widehat{P}_z^2 + 1\right]^{1/2}. \quad (50)$$

This is a time-dependent system with only two degrees of freedom. As $\widehat{P}_z$ is an obvious first integral, one can evacuate the conjugate variable $\widehat{z}$ and say, even if it is not academic, that we have a time-dependent system with one degree of freedom.

Let us perform now the following canonical transformation

$$P_1 = P - a\cos\widehat{t}, \quad Q_1 = Q, \quad (51)$$

generated by

$$F_2\left(Q_1, \widehat{z}, P, \widehat{P}_z\right) = Q_1 P + \widehat{z}\widehat{P}_z - aQ_1\cos\widehat{t}, \quad (52)$$

The Hamiltonian in terms of the new variables is

$$\widetilde{H} = \left[P^2 + \Omega_0^2 Q^2 + \widehat{P}_z^2 + 1\right]^{1/2} + aQ\sin\widehat{t}. \quad (53)$$

This is still a time-dependent system with only two degrees of freedom. $\widehat{P}_z$ is still a constant of motion. The equations of Hamilton read

$$\dot{Q} = \frac{P}{\gamma}$$
$$\dot{P} = -\frac{\Omega_0^2 Q}{\gamma} - a\sin\widehat{t}. \quad (54)$$

This set of equations is similar to the one found by Kwon and Lee to describe the motion of a particle in a constant and homogeneous magnetic field and an oscillating electric field of arbitrary polarization [29].

These equations of motion are solved numerically. We have assumed that $\widehat{P}_z = 0$ in every case. Chaos is evidenced first by performing Poincaré maps. The plane $P - Q$ with $\widehat{t} = 0$ (mod2π) is chosen to be the Poincaré surface of section. Figure 2a shows Poincaré maps for only one trajectory. The Lyapunov exponent for this trajectory has also been calculated by using Benettin's method [9,10,20]. To do so, a very close trajectory is considered, the very small distance between the initial distance is d_0. A sequence d_n corresponding to these trajectories is calculated numerically. For every fixed time Δt, or for every fixed distance ratio d/d_0, d_n is renormalized to d_0. The two ways to renormalize are used and compared, Figure 2b shows the good agreement obtained for the Lyapunov exponents when using the two renormalization techniques. The fact that we have chaotic trajectories shows that the system is not integrable.

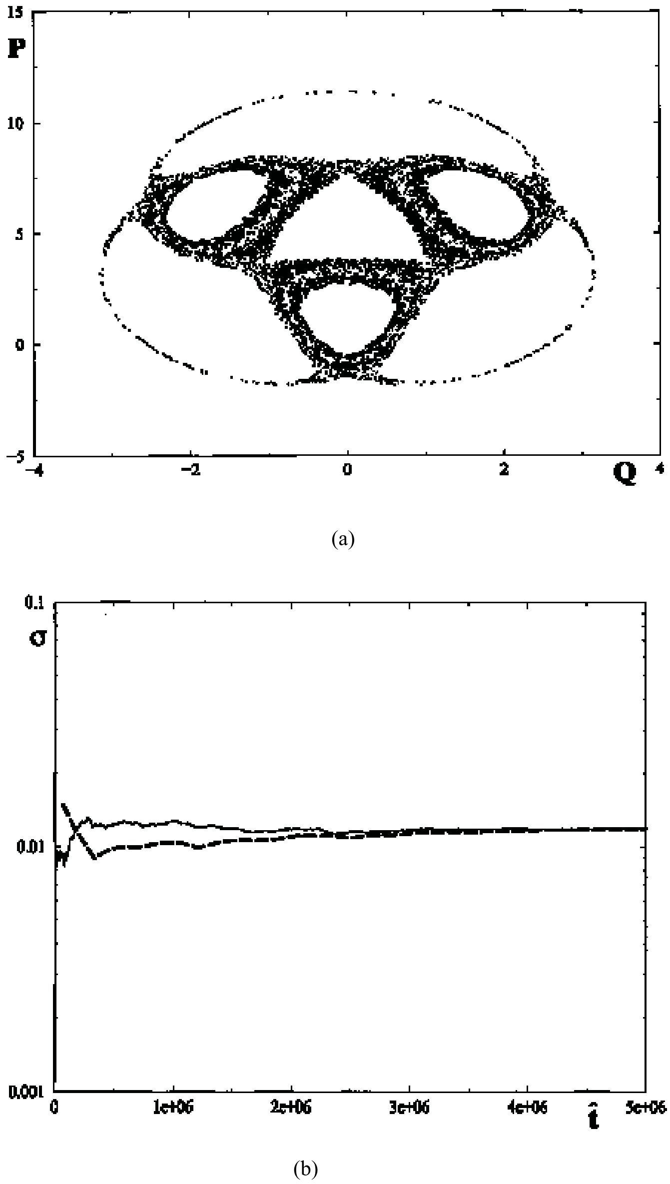


Fig. 2. $a = 4.03$, $\Omega_0 = 2$. (a) Surface of section plots for one trajectory. (b) Lyapunov exponent calculated with the same trajectory, the two renormalization methods are compared ($a = 4.03$, $\Omega_0 = 2$).

Performing Poincaré maps, one can check that the primary resonance (3,1) exists for all non-zero values of the dimensionless electric field a while the secondary (3,1) resonance appears when a is greater that some threshold value of a for one given value of Ω_0. When $\Omega_0 = 2$ the secondary resonance is born when a is in the range 3–3.33. As a is increased, the central island is squeezed by the hyperbolic fixed points of the secondary (3,1) resonance until the fixed points are absorbed by the elliptic fixed point of the (1,1) resonance. When a is increased, the hyperbolic fixed points reappear and move outward. When a is in the range 3.38–4, the primary (3,1) resonance and the secondary (3,1) resonance overlap, and chaotic trajectories cover the overlapped region.

When going back to the laboratory frame, in the case when the index of refraction is very close to unity, performing Poincaré maps and calculating Lyapunov exponents evidenced chaotic trajectories.

It has been shown in this paragraph that, as soon has the phase velocity is higher than the speed of light, chaotic trajectories can exist due to the overlap of only two resonances. Then some particles can move in a larger phase space and take more energy to the wave than in the integrable case. Still, the new space is not much larger as we have two resonances only. The situation studied next is more interesting as their will be an infinite number of resonances, thus a very large space will possibly be opened to the charged particle.

4 Dynamics of a charged particle in two or three linearly polarized traveling waves

In this part, the stability of a charged particle in two or three linearly polarized waves is studied. One of the waves is assumed to have an ultra-high intensity. The second and third perturbing wave can represent other laser beams or plasma waves.

When the two waves propagate in the same direction and have the same phase velocity ($V = \omega_0/k_0 = \omega_1/k_1$), one still has $\dot{P}_z = -\partial H/\partial z = (1/V)dH/dt$. Thus, $C = H - VP_z$, is a first integral of the system, and, consequently, the problem is integrable. When, $\omega_0/k_0 \neq \omega_1/k_1$, we have shown numerically that trajectories can become chaotic. When the directions of propagation of the two waves are different, the problem is not integrable. Chaotic trajectories were also evidenced.

In this chapter, we focus on the case of waves propagating in different directions and assume that the second and third waves are perturbing transverse electromagnetic waves or longitudinal perturbations.

4.1 All the waves are transverse

Let us consider a transverse perturbation polarized perpendicularly to the polarization plane of the high intensity wave. The wave vector of the perturbing mode with vector potential $\mathbf{A}_1$ is assumed to be at some angle α with respect to the wave vector of the high intensity wave, one has

$$\mathbf{A}_1 = A_1 \mathbf{e}_y \sin\left(\omega_1 t - k_{1\parallel} z - k_{1\perp} x\right). \tag{55}$$

In the extended phase space, when using the proper time of the particle to parametrize the motion, the Hamiltonian of the particle can be put in the following form [22,24]

$$H = \frac{1}{2}mc^2\gamma^2 - \frac{1}{2m}\left(\mathbf{P} + e\mathbf{A} + e\mathbf{A}_1\right)^2 - \frac{1}{2}mc^2, \tag{56}$$

or, neglecting terms in $\mathbf{A}_1^2$

$$H = H_0 + H_1 = H_0 - \frac{1}{m}(\mathbf{P} + e\mathbf{A}) \cdot e\mathbf{A}_1, \tag{57}$$

where H_0 is the Hamiltonian of the system when the perturbation, H_1, is neglected. Considering a high intensity plane wave, we know that this problem is integrable. We have: $H_0 = (1/2)mc^2\gamma^2 - (1/2m)(\mathbf{P} + e\mathbf{A})^2 - (1/2)mc^2$, and chose: $\mathbf{A} = (E_0/\omega_0)\cos(\omega_0 t - k_0 z)\widehat{\mathbf{e}}_x$.

The dimensionless variable $\widehat{x} = k_0 x$ is added to the dimensionless variables and parameter defined by equations (3). They are used to normalize the equations of motion. A normalized vector potential for the perturbation $\mathbf{A}_1$, $\mathbf{a}_1 = e\mathbf{A}_1/mc = e\mathbf{E}_1/mc\omega_1$, and a normalized proper time, $\widehat{\tau} = \omega_0\tau$, are also introduced. The normalized Hamiltonian is given by

$$\begin{aligned}\widehat{H} &= \widehat{H}_0 + \widehat{H}_1,\\ &= \frac{1}{2}\gamma^2 - \frac{1}{2}\left(\widehat{\mathbf{P}} + \mathbf{a} + \mathbf{a}_1\right)^2 - \frac{1}{2},\\ &= \widehat{H}_0 - \widehat{P}_y a_1,\end{aligned} \tag{58}$$

where: $\widehat{H}_0 = (1/2)\gamma^2 - (1/2)(\widehat{\mathbf{P}} + \mathbf{a})^2 - 1/2$ and $\mathbf{a}_1 = a_1\mathbf{e}_y \sin(\widetilde{\omega}_1\widehat{t} - \tilde{k}_{1\parallel}\widehat{z} - \tilde{k}_{1\perp}\widehat{x})$, with $\widetilde{\omega}_1 = \omega_1/\omega_0$, $\tilde{k}_{1\parallel} = k_{1\parallel}/k_0$ and $\tilde{k}_{1\perp} = k_{1\perp}/k_0$. We have assumed that $\widehat{P}_y a_1 \gg a_1^2$, which means that we consider that $\widehat{P}_y \approx 1$ and $a_1 \ll 1$.

First, the phase, $\kappa = \widetilde{\omega}_1\widehat{t} - \tilde{k}_{1\parallel}\widehat{z} - \widehat{k}_{1\perp}\widehat{x}$, is expressed in terms of the coordinates introduced by the canonical transformation defined by equations (10). We have

$$\kappa = \kappa' - \delta\sin\zeta - \beta\sin 2\zeta, \tag{59}$$

with

$$\begin{aligned}\kappa' &= -\widetilde{\omega}_1\phi - \tilde{k}_{1\parallel}\varphi - \tilde{k}_{1\perp}\theta,\\ \delta &= \frac{aP_\perp\left(\tilde{k}_{1\parallel} - \widetilde{\omega}_1\right)}{\left(P_\parallel - E\right)^2} - \frac{a\tilde{k}_\perp}{P_\parallel - E},\\ \beta &= \frac{a^2\left(\tilde{k}_{1\parallel} - \widetilde{\omega}_1\right)}{8\left(P_\parallel - E\right)^2}.\end{aligned} \tag{60}$$

Using the identities $\cos(u\sin v) = \sum_n J_n(u)\cos(nv)$ and $\sin(u\sin v) = \sum_m J_m(u)\sin(mv)$, one obtains

$$\sin\kappa = \sum_{n,m} J_n(\delta)J_m(\beta)\sin\left[\kappa' - (n+2m)\zeta\right]. \tag{61}$$

Consequently, $\widehat{H}_1$ reads in terms of the new coordinates

$$\begin{aligned}\widetilde{H}_1 &= -\widehat{p}_y a_1 \sin\kappa\\ &= -\widehat{p}_y a_1 \sum_{n,m} J_n(\delta)J_m(\beta)\sin\left[\kappa' - (n+2m)\zeta\right].\end{aligned} \tag{62}$$

Then

$$\begin{aligned}\widetilde{H}\left(P_\parallel, P_\perp, \widehat{P}_y, E, \theta, \varphi, \phi\right) =& -\frac{1}{2}\left(M^2 + P_\parallel^2 + P_\perp^2 + \widehat{P}_y^2 - E^2\right)\\ &- \widehat{P}_y a_1 \sum_{n,m} J_n(\delta)J_m(\beta)\sin\left[\kappa' - (n+2m)\zeta\right].\end{aligned} \tag{63}$$

The generalized Bessel function is usually defined as $C_n(\delta,\beta) = \sum_{j=-\infty}^{j=+\infty} J_{N+2j}(\delta)J_j(\beta)$ is introduced [30]. Letting $N = (n+2m)$, one obtains

$$\widetilde{H}_1 = -\widehat{P}_y a_1 \sum_N V_N \sin(\kappa' - N\zeta), \tag{64}$$

with

$$V_N(\delta,\beta) = (-1)^N C_{-N}(\delta,\beta). \tag{65}$$

The following symmetry relation [31]

$$C_{-N}(\delta,\beta) = (-1)^{-N} C_N(\delta,-\beta), \tag{66}$$

leads to

$$V_N(\delta,\beta) = C_N(\delta,-\beta). \tag{67}$$

The Hamiltonian can be expressed as a sum of harmonic interactions

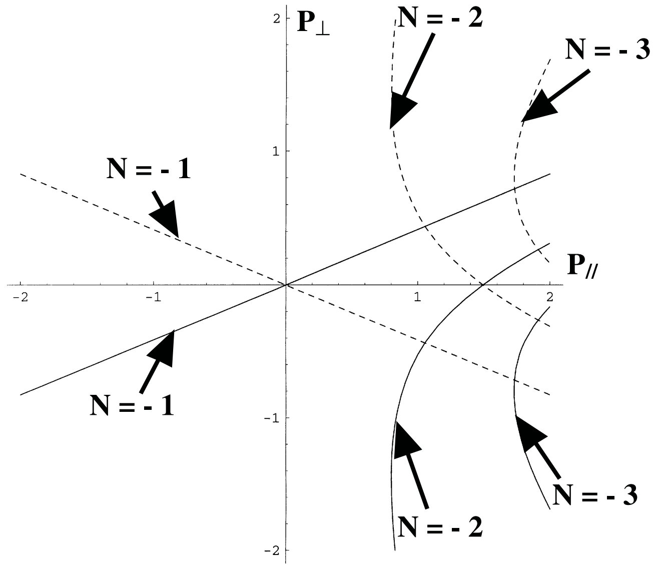
$$\widetilde{H}\left(P_{\|}, P_{\perp}, \widehat{P}_y, E, \theta, \varphi, \phi\right) = -\frac{1}{2}\left(M^2 + P_{\|}^2 + P_{\perp}^2 + \widehat{P}_y^2 - E^2\right) + a_1 \widehat{P}_y \sum_N V_N \sin\left[\tilde{k}_{1\|}\varphi + \tilde{k}_{1\perp}\theta + \widetilde{\omega}_1\phi + N(\varphi+\phi)\right]. \tag{68}$$

When the zero-order solution (13) is plugged in the argument of the perturbation sines, the stationary phase condition leads to the following resonance condition [24]

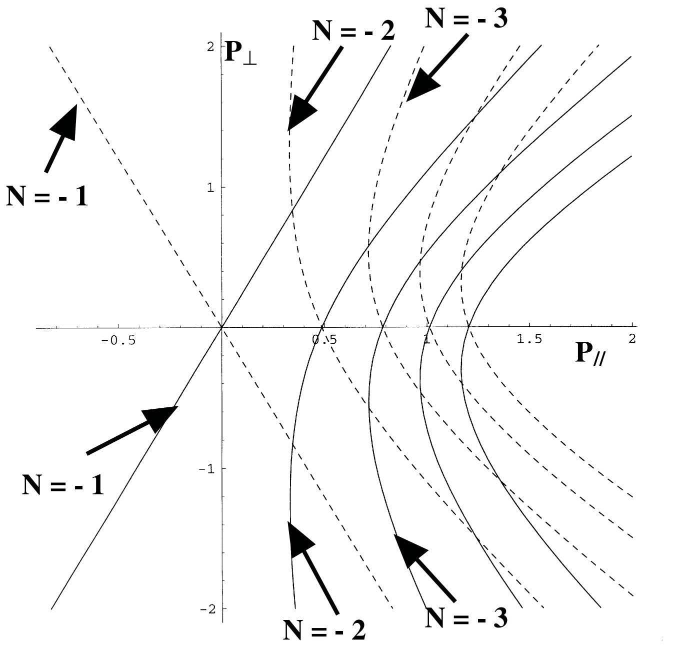
$$\tilde{k}_{1\|}P_{\|} + \tilde{k}_{1\perp}P_{\perp} - \widetilde{\omega}_1 E - N\left(E - P_{\|}\right) = 0. \tag{69}$$

This resonance condition when restricted to the energy surface equation given by the following expression: $E(P_{\perp}, \widehat{P}_y, P_{\|}, a) = \sqrt{M^2 + P_{\perp}^2 + \widehat{P}_y^2 + P_{\|}^2}$ gives $P_{\perp}$ versus $P_{\|}$. We have plotted these resonances in the case when $\widehat{P}_y = 0$ in (69). Considering that $\widehat{P}_y \neq 0$ comes out to assuming a higher value of a, $\tilde{a}$, given by $\tilde{a} = \sqrt{a^2 + 2\widehat{P}_y^2}$. When $\widehat{P}_y = 0$, $a = 1$, $\widetilde{\omega}_1 = 1$, $\tilde{k}_{1\|} = \sqrt{2}/2$, $\tilde{k}_{1\perp} = \pm\sqrt{2}/2$, and $N = -1, -2, -3$, Figures 3 show $P_{\perp}$ versus $P_{\|}$ when the resonance condition is satisfied, for $\alpha = \pi/4$ and $\alpha = 3\pi/4$. The solid lines are obtained when assuming that the perturbation is such that $\tilde{k}_{1\perp} = \sqrt{2}/2$. The dashed lines correspond to the situation when $\tilde{k}_{1\perp} = -\sqrt{2}/2$. Figures 3 show that the lines are quite far from each other when considering one perturbing wave only. One can conclude that a second perturbing wave is necessary so that many resonances overlap. As a consequence, two perturbing waves were considered in order to define conditions when efficient stochastic heating takes place. We consider that the two wave-vectors of the perturbation are symmetric with respect to the direction of propagation of the very intense wave.

Figures 4 are devoted to the case when $\widetilde{\omega}_1 = 3$. The whole numerical results show that the resonance overlap seems to be easier (as the resonances get closer to each other) when α and $\widetilde{\omega}_1$ go to higher values.



(a)

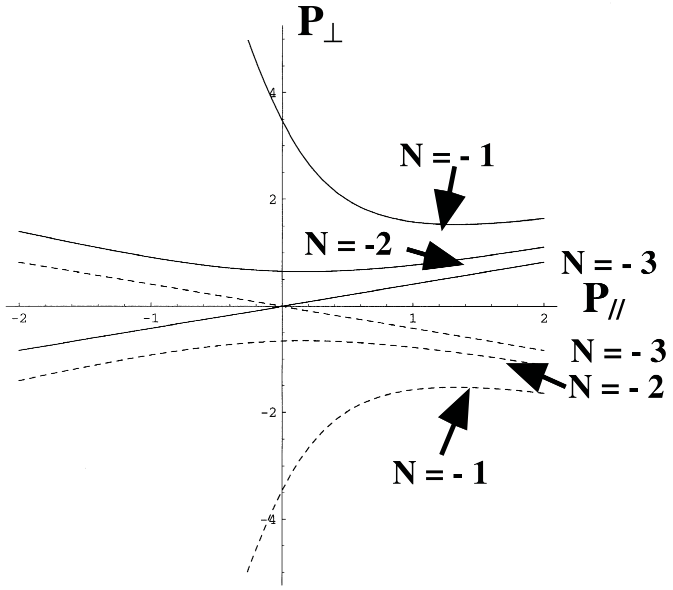


(b)

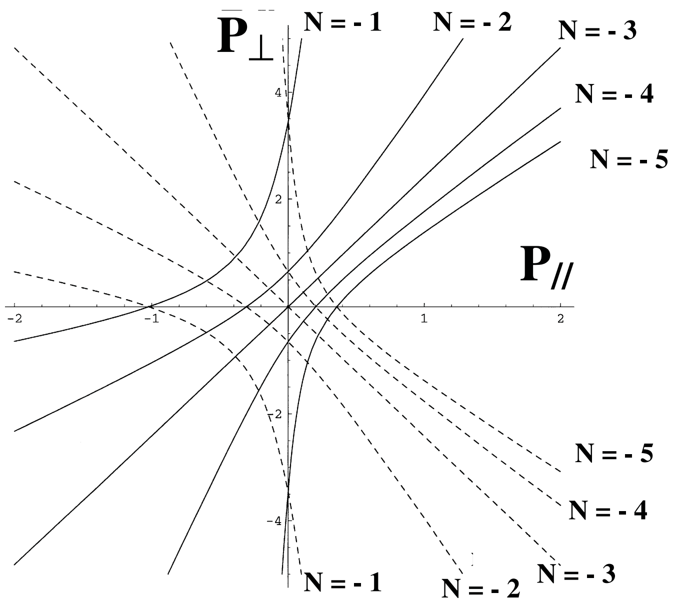
Fig. 3. Resonances in the $(P_{\|}, P_{\perp})$ plane, $a = 1$, $|\tilde{k}_{1\|}| = |\tilde{k}_{1\perp}| = \sqrt{2}/2$, $\widetilde{\omega}_1 = 1$. (a) $\alpha = \pi/4$, (b) $\alpha = 3\pi/4$.

In order to calculate the width of the Nth resonance, following Rax [24], a resonant torus $(P_{\|c}, P_{\perp c}, E_c)$ verifying the resonance condition for a certain value of N is isolated. Then, the following new variables are introduced

$$J = \frac{P_{\|} - P_{\|c}}{\tilde{k}_{1\|} + N} = \frac{P_{\perp} - P_{\perp c}}{\tilde{k}_{1\perp}} = \frac{E - E_c}{\widetilde{\omega}_1 + N},$$
$$\psi = \tilde{k}_{1\|}\varphi + \tilde{k}_{1\perp}\theta + \widetilde{\omega}_1\phi + N(\varphi+\phi). \tag{70}$$



(a)



(b)

Fig. 4. Resonances in the $(P_\|, P_\perp)$ plane, $a = 1$. $|\tilde{k}_{1\|}| = |\tilde{k}_{1\perp}| = \sqrt{2}/2$, $\widetilde{\omega}_1 = 3$. (a) $\alpha = \pi/4$, (b) $\alpha = 3\pi/4$.

Then, the perturbed motion is described by a one-dimensional oscillator,

$$\begin{aligned}\frac{dJ}{d\tau} &= -a_1 \widehat{P}_y V_N \left(P_{\perp c}, P_{\| c}, E_c\right)\cos\psi,\\ \frac{d\psi}{d\tau} &= -\left[\left(\tilde{k}_{1\|}+N\right)^2 + \tilde{k}_{1\perp}^2 - (\widetilde{\omega}_1+N)^2\right] J. \end{aligned} \tag{71}$$

The half-width of this resonance is [6,10]

$$\Delta J = 2\sqrt{\frac{a_1\left|\widehat{P}_y\right| |V_N|}{\left|\tilde{k}_{1\perp}^2 + \left(\tilde{k}_{1\|}-\widetilde{\omega}_1\right)\left(\tilde{k}_{1\|}+\widetilde{\omega}_1+2N\right)\right|}}. \tag{72}$$

Consequently, the half-width in terms of the action variables is given by

$$\begin{aligned}\Delta P_\| &= \left(\tilde{k}_{1\|}+N\right)\Delta J,\\ \Delta P_\perp &= \tilde{k}_{1\perp}\Delta J.\end{aligned} \tag{73}$$

The sum of the half-widths of the two resonances N and N', corresponding to modes with wave vectors $\tilde{k}_\perp$ and $-\tilde{k}_\perp$ is given by

$$\begin{aligned}\Delta P_{N\perp} + \Delta P_{N'\perp} &= 2\left|\tilde{k}_{1\perp}\right|\left(\left|a_1\widehat{P}_y\right|\right)^{\frac{1}{2}}\\ &\times\left[\left|\frac{V_{N,\tilde{k}_{1\perp}}}{\tilde{k}_{1\perp}^2+\left(\tilde{k}_{1\|}-\widetilde{\omega}_1\right)\left(\tilde{k}_{1\|}+\widetilde{\omega}_1+2N\right)}\right|^{\frac{1}{2}}\right.\\ &\left.+\left|\frac{V_{N',-\tilde{k}_{1\perp}}}{\tilde{k}_{1\perp}^2+\left(\tilde{k}_{1\|}-\widetilde{\omega}_1\right)\left(\tilde{k}_{1\|}+\widetilde{\omega}_1+2N'\right)}\right|^{\frac{1}{2}}\right],\end{aligned} \tag{74a}$$

and

$$\Delta P_{N\|} + \Delta P_{N'\|} = \left|\frac{\tilde{k}_{1\|}+N}{\tilde{k}_{1\perp}}\right|\Delta P_{N\perp} + \left|\frac{\tilde{k}_{1\|}+N'}{\tilde{k}_{1\perp}}\right|\Delta P_{N'\perp}. \tag{74b}$$

Let us consider the overlap of resonances $N = -1$ corresponding to the two modes with wave vector components $\tilde{k}_{1\perp}$ and $-\tilde{k}_{1\perp}$ when $\widetilde{\omega}_1 = 1$. The quantities $V_{-1,k_\perp}$ and $V_{-1,-k_\perp}$ are evaluated by using the following Taylor expansion relevant for small α and β [24,32,33]

$$C_1(\delta,\beta) = \frac{\delta}{2} + \frac{\delta\beta}{4} - \frac{\delta^3}{16}. \tag{75}$$

In the situations considered here, the results obtained when using this expansion are in good agreement with those obtained through a numerical evaluation of the series representation of this generalized Bessel function. The Chirikov criterion [21] is satisfied for two resonances when their unperturbed separatrices touch or overlap. When they are close enough, a trajectory is no longer locked within one of the resonances, and it can pass from one resonance to the other. In other words the Chirikov threshold is reached when $\Delta P_{-1\perp} + \Delta P'_{-1\perp}$ and $\Delta P_{-1\|} + \Delta P'_{-1\|}$ are larger than the distance between the two resonances in term of $P_\perp$ and $P_\|$ respectively. This comes out to verify the two following conditions

$$R_{-1\perp} = \frac{d_{-1\perp}}{\Delta P_{-1\perp} + \Delta P'_{-1\perp}} \le 1, \tag{76a}$$

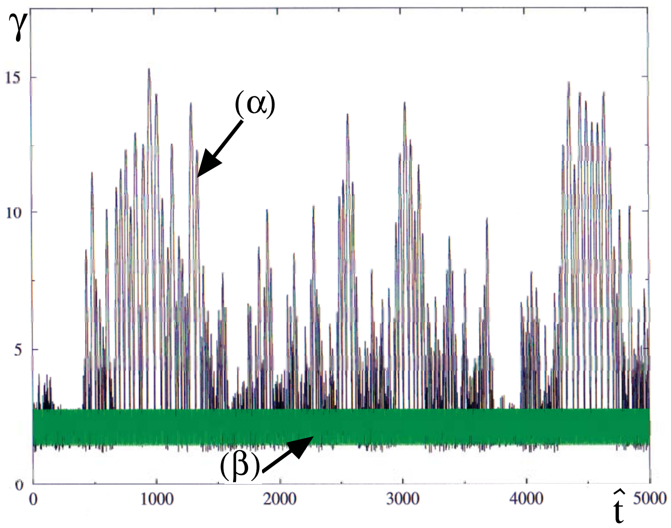


Fig. 5. Normalized energy of the charged particle versus time. $\widehat{P}_y = 1$, $\widetilde{\omega}_1 = 3$, $\alpha = \pi/3$. (α) $a = 2$, $a_1 = 0.2$, (β) the total intensity is in one wave: $a = 2.176$, $a_1 = 0$.

and

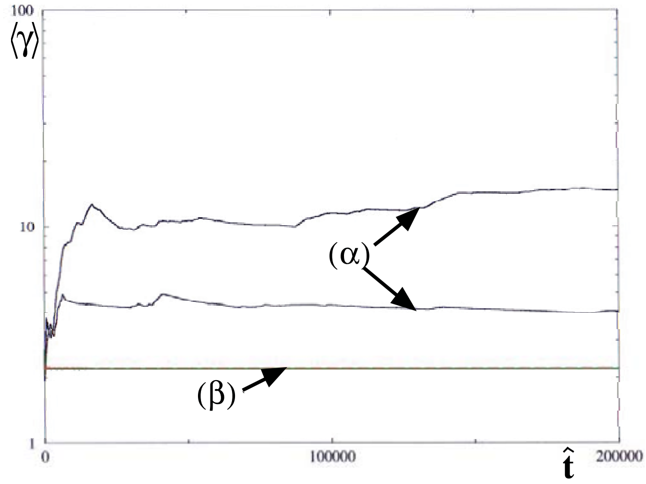
$$R_{-1\parallel} = \frac{d_{-1\parallel}}{\Delta P_{-1\parallel} + \Delta P'_{-1\parallel}} \leq 1. \tag{76b}$$

where $d_{-1\perp}$ and $d_{-1\parallel}$ are the distance between the two resonances respectively along the $P_\perp$-axis and the $P_\parallel$-axis.

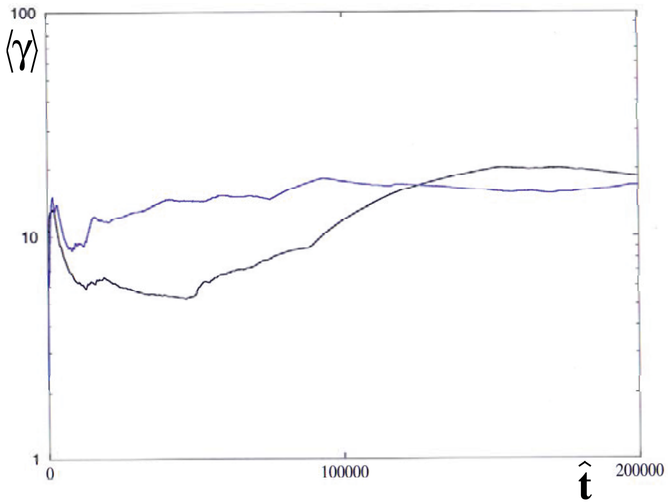
The values of $P_\parallel$ and $P_\perp$ which are used in our numerical estimates verify the resonance condition (69). Our numerical estimates shows that there is stochastic heating for realistic laser parameters. For instance when $a = 1$, $a_1 = 0.1$, $\widehat{P}_y = 1$ and $\alpha = \pi/4$, considering two resonances located at $P_\parallel = 0.4$ and $P_\perp = \pm 0.165$, we have $R_{-1\perp} = 0.55$ and $R_{-1\parallel} = 0$. In this case, the Chirikov threshold is reached. Considering that $a = 4$, $a_1 = 0.3$, $\widehat{P}_y = 1$ and $\alpha = \pi/3$, and the same two resonances, $P_\parallel = 0.4$ and $P_\perp = \pm 0.231$, we have $R_{-1\perp} \approx 0.33$ and $R_{-1\parallel} = 0$, which means that, in this case, the Chirikov criterion is better satisfied. Considering a trajectory with its initial conditions in the overlap region, the energy of one particle in the three waves is compared to the one when the total electromagnetic intensity is in the high intensity wave. Only weak stochastic heating seems to occur in this case.

When assuming that $\widetilde{\omega}_1 = 3$, the resonance overlap seems to be easier to achieve. Then, the most dangerous resonance is $N = -3$ (Figs. 4). In order to calculate the resonance half-widths of each resonance, the generalized Bessel function, $C_3(\delta, \beta)$, is estimated through its series representation. Just like in the previous case, considering $N = -3$, it can be shown that the Chirikov condition is satisfied when

$$R_{-3\perp} = \frac{d_{-3\perp}}{\Delta P_{-3\perp} + \Delta P'_{-3\perp}} \leq 1, \tag{77a}$$



(a)



(b)

Fig. 6. Average value of the normalized energy versus time. $\widehat{P}_y = 1$, $\widetilde{\omega}_1 = 3$, $\alpha = \pi/3$. (a)(α) three waves and two initial conditions are considered: $a = 2$, $a_1 = 0.2$. (β) considering the two same initial conditions, the total intensity is assumed to be in the high intensity wave: $a = 2.176$, $a_1 = 0$. (b) $a = 2$, $a_1 = 0.4$, the same two initial conditions are considered.

and

$$R_{-3\parallel} = \frac{d_{-3\parallel}}{\Delta P_{-3\parallel} + \Delta P'_{-3\parallel}} \leq 1. \tag{77b}$$

When $a = 2$, $a_1 = 0.2$, $\widehat{P}_y = 1$ and $\alpha = \pi/3$, considering $P_\parallel = 0.4$ and $P_\perp = \pm 0.231$, we have $R_{-3\perp} = 0.54$ and $R_{-3\parallel} = 0$. In this case, we are well above the Chirikov threshold for the resonance $N = -3$. Figure 5 shows the evolution of the energy of one particle in the case when it interacts with the three modes and in the case when all the intensity is in one mode. This figure shows that there is stochastic heating. Figure 6a shows the evolution of the average value of the energy versus time for two distinct initial conditions chosen in the overlap region. Because of stochastic heating, the average energy for each initial condition is higher in the two chaotic cases than in the integrable ones. In the two integrable cases the average values of the energy remain different. The comparison between the integrable case and the chaotic one shows that one has stochastic heating. For the same conditions but a higher value of a_1, $a_1 = 0.4$, the two curves converge approximately with time in the cases when three waves are considered (Fig. 6b). This is a consequence of the fact that the initial conditions are in a subspace that becomes ergodic.

4.2 The perturbing wave is longitudinal

The wave vector of each longitudinal perturbing mode is also assumed to propagate at some angle α with respect to the wave vector of the high intensity wave.

First, only one perturbing wave is considered. In the extended phase space, the Hamiltonian of the charged particle in the field of a high intensity wave and one low intensity longitudinal mode is given by [22]

$$H = \frac{1}{2m}\left(\frac{W}{c} + e\frac{\delta\phi_1}{c}\right)^2 - \frac{1}{2m}(\mathbf{P} + \mathbf{eA})^2 - \frac{1}{2}mc^2, \quad (78)$$

where $\delta\phi_1 = \overline{\delta\phi}_1 \sin(\omega_1 t - k_{1\perp} y - k_{1\|} z)$ is the scalar potential of the perturbing wave and W is the total energy of the particle, W is given by

$$W = \left[\left[P_x + \frac{eE_0}{\omega_0}\cos(\omega_0 t - k_0 z)\right]^2 c^2 + \left(P_y^2\right)c^2 + \left(P_z^2\right)c^2 + m^2c^4\right]^{1/2} - e\overline{\delta\phi}_1 \sin\left(\omega_1 t - k_{1\|} z - k_{1\perp} x\right). \quad (79)$$

Neglecting terms in $\delta\phi_1^2$, the Hamiltonian becomes

$$H = H_0 + \gamma e \delta\phi_1. \quad (80)$$

The wave vector $\mathbf{A}$ is still given by: $\mathbf{A} = (E_0/\omega_0)\cos(\omega_0 t - k_0 z)\widehat{\mathbf{e}}_x$. H_0 is the zero-order Hamiltonian and the perturbation term is: $H_1 = \gamma(e\delta\phi_1/mc^2)$.

Once more, this Hamiltonian can be normalized. The following Hamiltonian gives the normalized equations of motion

$$\widehat{H} = \widehat{H}_0 + \gamma \widehat{\delta E}_p, \quad (81)$$

where $\widehat{\delta E}_p = e\delta\phi_1/mc^2$ is the normalized potential energy. In terms of the variables defined by (10) the perturbation reads

$$\widehat{H}_1 = \overline{\widehat{\delta E}}_p \left[E - \frac{P_\perp a}{P_\| - E}\cos\zeta - \frac{a^2}{4\left(P_\| - E\right)}\cos 2\zeta\right] \times \left\{\sum_{n,m} J_n(\delta) J_m(\beta) \sin\left[\kappa' - (n+2m)\zeta\right]\right\}, \quad (82)$$

where $\overline{\widehat{\delta E}}_p$ is the amplitude of the potential oscillation $\widehat{\delta E}_p$. The perturbation can be brought in the following simple form

$$\widetilde{H}_1 = \overline{\widehat{\delta E}}_p \sum_{h,m,n} J_n(A) J_m(B) U_h \sin\left[\kappa' - (n+2m+h)\zeta\right], \quad (83)$$

with

$$U_h = E\delta_h^0 - \frac{P_\perp a}{2\left(P_\| - E\right)}\delta_{|h|}^1 - \frac{a^2}{8\left(P_\| - E\right)}\delta_{|h|}^2, \quad (84)$$

where the h sum is restricted to $h = 0, \pm 1, \pm 2$, δ_j^i are the Kronecker symbols. Introducing the generalized Bessel function, we obtain

$$\widetilde{H}_1 = -\overline{\widehat{\delta E}}_p \sum_N V_N(\delta,\beta) \sin\left[\tilde{k}_{1\|}\varphi + \tilde{k}_{1\perp}\theta + \widetilde{\omega}_1\phi + N(\varphi + \phi)\right], \quad (85)$$

with

$$V_N(\delta,\beta) = \sum_{|h|=0,1,2} U_h C_{N-h}(\delta,-\beta), \quad (86)$$

that is to say

$$V_N(\delta,\beta) = EC_N(\delta,-\beta) - \frac{P_\perp a}{2\left(P_\| - E\right)}\left[C_{N+1}(\delta,-\beta) + C_{N-1}(\delta,-\beta)\right] - \frac{a^2}{8\left(P_\| - E\right)}\left[C_{N+2}(\delta,-\beta) + C_{N-2}(\delta,-\beta)\right]. \quad (87)$$

Plugging the zero-order solution (13) in the perturbation leads to the resonance condition given by (69).

The Chirikov threshold criterion [21] is fulfilled for resonances N and N', corresponding to two modes with the same wave vectors $\tilde{k}_\perp$ when

$$R_{N,N'_\perp} = \frac{d_{N,N'_\perp}}{\Delta P_\perp + \Delta P'_\perp} \le 1, \quad (88)$$

where $d_{N,N'\perp}$ is the distance along the $P_\perp$ axis between the two resonances and where

$$\Delta P_\perp + \Delta P'_\perp = 2\left|\tilde{k}_{1\perp}\right| \left[\left(\frac{\overline{\widehat{\delta E}}_p |V_N|}{\left|\tilde{k}_{1\perp}^2 + \left(\tilde{k}_{1\|} - \widetilde{\omega}_1\right)\left(\tilde{k}_{1\|} + \widetilde{\omega}_1 + 2N\right)\right|}\right)^{\frac{1}{2}} + \left(\frac{\overline{\widehat{\delta E}}_p |V_{N'}|}{\left|\tilde{k}_{1\perp}^2 + \left(\tilde{k}_{1\|} - \widetilde{\omega}_1\right)\left(\tilde{k}_{1\|} + \widetilde{\omega}_1 + 2N'\right)\right|}\right)^{\frac{1}{2}}\right], \quad (89)$$

and when

$$R_{N,N'\|} = \frac{d_{N,N'\|}}{\Delta P_\| + \Delta P'_\|} \le 1, \quad (90)$$

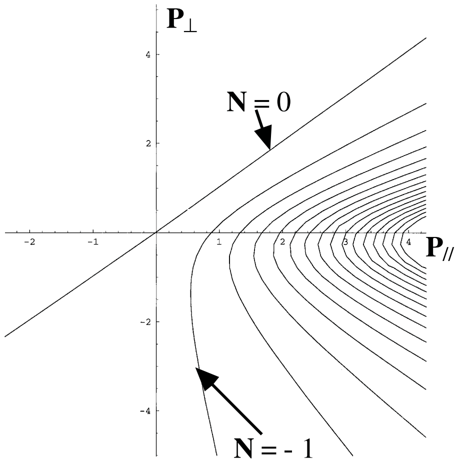


Fig. 7. Resonances in the $(P_{\|}, P_{\perp})$ plane. $a = 1$, $\widetilde{\omega}_1 = 1\times10^{-2}$, $\alpha = 3\pi/4$ and $\tilde{k}_1 = 1$.

with

$$\Delta P_{\|} + \Delta P'_{\|} = 2\left|\tilde{k}_{1\|} + N\right| \left[\frac{\widehat{\overline{\delta E}}_p\,|V_N|}{\left|\tilde{k}^2_{1\perp} + \left(\tilde{k}_{1\|} - \widetilde{\omega}_1\right)\left(\tilde{k}_{1\|} + \widetilde{\omega}_1 + 2N\right)\right|}\right]^{\frac{1}{2}} +2\left|\tilde{k}_{1\|} + N'\right| \left[\frac{\widehat{\overline{\delta E}}_p\,|V_{N'}|}{\left|\tilde{k}^2_{1\perp} + \left(\tilde{k}_{1\|} - \widetilde{\omega}_1\right)\left(\tilde{k}_{1\|} + \widetilde{\omega}_1 + 2N'\right)\right|}\right]^{\frac{1}{2}}. \quad (91)$$

The resonances in the $(P_{\|}, P_{\perp})$ plane are shown in Figure 7 for one value of $\tilde{k}_1$.

In order to compute the resonance half-width of each resonance, (75) and the following Taylor expansions were used [24,32,33]

$$C_0(\delta,\beta) = 1 - \frac{\delta^2}{4} + \frac{\delta^4}{64} - \frac{\beta^2}{4},$$
$$C_2(\delta,\beta) = \frac{\delta^2}{8} + \frac{\delta^2\beta}{8} - \frac{\delta^4}{96} - \frac{\beta}{2},$$
$$C_3(\delta,\beta) = -\frac{\delta\beta}{4} + \frac{\delta^3}{48},$$
$$C_4(\delta,\beta) = \frac{\delta^4}{384} - \frac{\delta^2\beta}{16} + \frac{\beta^2}{8}. \quad (92)$$

In the following discussion, the results obtained when using these Taylor expansions are in good agreement with the evaluations derived through the series representations of these functions.

Considering two resonances $N = -1$ and $N = -2$, when $\tilde{k}_1 = 1$ and $\widehat{\overline{\delta E}}_p = 0.1$ ($\tilde{k}_1\widehat{\overline{\delta E}}_p = 0.1$), we have assumed $P_{\perp} = 0$, $P_{\|} = 0.871$ for the resonance $N = -1$ and $P_{\|} = 1.327$ for $N = -2$. The following two values $R_{N,N'\perp} = 0$ and $R_{N,N'\|} = 0.37$, were obtained. As a consequence the Chirikov criterion is satisfied for these two resonances. No strong stochastic heating was observed when considering trajectories having their initial conditions in the overlap region.

Then, in order to observe strong stochastic heating we have considered two symmetric perturbing longitudinal waves. The two electric fields of the perturbing waves are supposed to be in the $(\widehat{x}, \widehat{z})$ plane. In this case many resonances should overlap. Let us consider the overlap of resonances N corresponding to the two modes with wave vector components $\tilde{k}_{1\perp}$ and $-\tilde{k}_{1\perp}$. The Chirikov criterion is satisfied when

$$R_{N\perp} = \frac{d_{N\perp}}{\Delta P_{N\perp} + \Delta P'_{N\perp}} \le 1, \quad (93)$$

with

$$\Delta P_{N\perp} + \Delta P'_{N\perp} = \frac{2\left|\tilde{k}_{1\|}\right|\left[\left|\widehat{\overline{\delta E}}_p V_{N,k_{1\perp}}\right|^{\frac{1}{2}} + \left|\widehat{\overline{\delta E}}_p V_{N,-k_{1\perp}}\right|^{\frac{1}{2}}\right]}{\left|\tilde{k}^2_{1\perp} + \left(\tilde{k}_{1\|} - \widetilde{\omega}_1\right)\left(\tilde{k}_{1\|} + \widetilde{\omega}_1 + 2N\right)\right|^{\frac{1}{2}}}. \quad (94)$$

and when

$$R_{N\|} = \frac{d_{N\|}}{\Delta P_{N\|} + \Delta P'_{N\|}} \le 1, \quad (95)$$

with

$$\Delta P_{N\|} + \Delta P'_{N\|} = \frac{2\left|\tilde{k}_{1\|} + N\right|\left[\left|\widehat{\overline{\delta E}}_p V_{N,k_{1\perp}}\right|^{\frac{1}{2}} + \left|\widehat{\overline{\delta E}}_p V_{N,-k_{1\perp}}\right|^{\frac{1}{2}}\right]}{\left|\tilde{k}^2_{1\perp} + \left(\tilde{k}_{1\|} - \widetilde{\omega}_1\right)\left(\tilde{k}_{1\|} + \widetilde{\omega}_1 + 2N\right)\right|^{\frac{1}{2}}}. \quad (96)$$

Considering two resonances $N = -1$ which are symmetric with respect to the $P_{\|}$-axis. Their coordinates are $P_{\|} = 0.8$ and $P_{\perp} = \pm0.124$. Assuming that $a = 1$, $\widetilde{\omega}_1 = 1 \times 10^{-2}$, $\tilde{k}_1 = 1$, $\alpha = 3\pi/4$, $\widehat{\overline{\delta E}}_p = 0.1$, we found $R_{N\perp} = 0.54$ and $R_{N\|} = 0$. The Chirikov criterion is satisfied for these two resonances. Figure 8 shows that stochastic heating can be observed considering one trajectory with its initial conditions between the two resonances

5 Conclusions

The stability of a charged particle in a high intensity linearly polarized traveling wave (above 10^{18} W/cm^2) was investigated within the framework of a Hamiltonian analysis.

The dynamics of one particle in a linearly polarized traveling wave propagating in a nonmagnetized medium was studied first. Integrability was demonstrated. It has been shown that the charged particle can have a high average velocity along the direction of propagation of the wave and along its electric field. These results were used to compare this one particle model to a plasma approach. The results obtained with the two approaches are in good

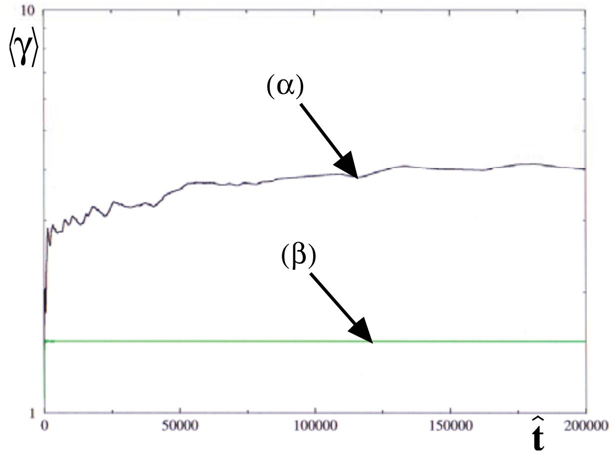


Fig. 8. Average value of the energy of the particle versus time $a = 1$, $\widetilde{\omega}_1 = 1 \times 10^{-2}$, $\tilde{k}_1 = 1$, $\alpha = 3\pi/4$; (a) in the case of the three waves, (b) when it interacts with the high intensity wave only.

agreement during a very short time. Our numerical results have shown that the plasma response has to be taken into account after this short time. It was shown that, even when the plasma response is taken into account, the motion of one particle of the plasma remains integrable in the limit of low densities and/or high intensities.

The dynamics of a charged particle in a linearly polarized electromagnetic wave propagating along a constant homogeneous magnetic field was investigated next. In a first part, the electromagnetic wave was assumed to propagate in vacuum. We have shown that the synchronous solution still exists when the wave is linearly polarized. Two constants, which are canonically conjugate, were used to reduce the problem to a two-degree of freedom problem. The system was integrated and was shown to be Liouville integrable. Then, in order to study the plasma response in the case of a very high intensity wave propagating in a low-density plasma, an approximate solution for an almost transverse wave was used. It is shown that for moderate values of the constant magnetic field one can consider that this wave has approximately the same form as the one that propagates in vacuum. Performing a simplifying Lorentz transformation eliminated the space variable corresponding to the direction of propagation of the wave. Just like in the previous case, two canonically conjugate constants were used to reduce the initially three degrees of freedom problem to a two degrees of freedom problem. Thus, in cases when the index of refraction is very close to unity, performing Poincaré maps evidenced chaotic trajectories. Lyapunov exponents were also calculated to confirm the chaotic nature of some trajectories. Chaos appears when a secondary resonance and a primary resonance overlap. As a consequence the system is not integrable and chaos appears as soon as the plasma response is taken into account.

Finally, the stability of a charged particle in the fields of several waves was explored. In this part no constant homogeneous magnetic field was considered. A high intensity plane wave was first perturbed by one or two electromagnetic plane waves. The effect of a perturbing plasma wave was also studied. The solution of Hamilton-Jacobi equation derived in Section 2 was used to identify resonances. The stochasticity threshold due to resonance overlap was calculated. The Chirikov criterion was applied to two resonances corresponding to two symmetric perturbing waves. Stochastic heating was evidenced considering trajectories and computing the energy or the average energy of the charged particle.

A next step will be to consider a high intensity wave propagating along a constant homogeneous magnetic field perturbed by an electromagnetic wave. The conjugate effect of the magnetic field might lead to closer resonance curves and more stochastic heating.

The authors thank Dr. Serge Bouquet for very valuable discussions.

Appendix A

Let us show that the system defined by equations (25) is Liouville integrable. To do so, let us consider the following Hamiltonian

$$\mathbf{H} = \frac{\bar{P}_1^2}{2} + \frac{\Omega_0^2}{2\widehat{C}^2} Q_1^2 - \frac{a}{\widehat{C}} Q_1 \sin\zeta. \tag{A.1}$$

Assuming that ζ plays the part of times, the equations of Hamilton read

$$\bar{P}_1' = -\frac{\partial \mathbf{H}}{\partial Q_1} = -\frac{\Omega_0^2}{\widehat{C}^2} Q_1 + \frac{a}{\widehat{C}} \sin\zeta.$$
$$Q_1' = \frac{\partial \mathbf{H}}{\partial \bar{P}_1} = \bar{P}_1. \tag{A.2}$$

Differentiating a second time the second equation, one finds again equation (29a) for Q_1. The following equation for $\bar{P}_1$, is obtained

$$\bar{P}_1'' + \frac{\Omega_0^2}{\widehat{C}^2} \bar{P}_1 = \frac{a}{\widehat{C}} \cos\zeta. \tag{A.3}$$

The equation for P_1 (29b) is found again by letting $\bar{P}_1 = -(P_1 + a\cos\zeta)/\widehat{C}$. Thus, one constant for the system defined by (A.1) will provide a constant for our system.

Writing the total derivative of the Hamiltonian (A.1) with respect to leads to

$$\frac{d\mathbf{H}}{d\zeta} + \frac{a}{\widehat{C}} Q_1 \cos\zeta = 0. \tag{A.4}$$

If one manages to find a function $f(Q_1, \bar{P}_1, \zeta)$ such as

$$\frac{df(Q_1, \bar{P}_1, \zeta)}{d\zeta} = \frac{a}{\widehat{C}} Q_1 \cos\zeta, \tag{A.5}$$

then the following quantity is a constant of motion

$$\mathbf{I} = \mathbf{H} + f(Q_1, \bar{P}_1, \zeta). \tag{A.6}$$

Performing two integrations by parts and using the equations of Hamilton, a function $f(Q_1, \bar{P}_1, \zeta)$ was found and the following constant of the system defined by (41) was obtained

$$\mathbf{K} = \left(\frac{\Omega_0^2}{\widehat{C}^2} - 1\right)\left(\frac{\bar{P}_1^2}{2} + \frac{\Omega_0^2}{2\widehat{C}^2}Q_1^2\right) - \frac{a}{\widehat{C}}\left(\frac{\Omega_0^2}{\widehat{C}^2}Q_1 \sin\zeta + \bar{P}_1 \cos\zeta + \frac{a}{4\widehat{C}}\cos 2\zeta\right). \quad (A.7)$$

which, in terms of variables P_1, Q_1 reads

$$\mathbf{K} = \left(\frac{\Omega_0^2}{\widehat{C}^2} - 1\right)\left[\frac{(P_1 + a\cos\zeta)^2}{2} + \frac{\Omega_0^2}{2}Q_1^2\right] - a\left[\frac{\Omega_0^2}{\widehat{C}^2}Q_1 \sin\zeta - (P_1 + a\cos\zeta)\cos\zeta + \frac{a}{4}\cos 2\zeta\right]. \quad (A.8)$$

The fact that this quantity is a constant of motion was verified numerically.

As a consequence, the system has two first integrals, the one that we have just found and the Hamiltonian. This shows again that the system is integrable.

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